Additive-State-Decomposition-Based Dynamic Inversion Stabilized Control for a Class of Uncertain Systems and Its Application to Missile Longitudinal Tracking

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Abstract

This paper presents a novel control, namely additive-state-decomposition-based dynamic inversion stabilized control, that is used to stabilize a class of systems subject to nonparametric time-varying uncertainties with respect to both state and input. By the additive state decomposition and a new definition of output, the uncertain system is transformed into an uncertainty-free system with relative degree one, in which all uncertainties are lumped into a new disturbance at output. Subsequently, dynamic inversion control is applied to reject the lumped disturbance. Performance analysis of the resulting closed-loop dynamics is given that the stability can be ensured if a key parameter is sufficiently small. Finally, this proposed control is applied to a tracking problem for the missile's angle of attack. Numerical simulation demonstrates its effectiveness.

I. INTRODUCTION

The stabilization problem for control systems subject to uncertainties with respect to both state and input is a challenging problem, which has become a subject of keen interest in recent years. It is well known that uncertainties at input are often present in practical systems, for example input with an uncertain gain, or nonlinearities inherently aroused from dead zone, quantization and backlash, etc. The existence of uncertain inputs is a source of degradation or instability in the performance of the system. In addition to uncertainties in the input, there exist plant uncertainties, which originate from various sources, such as variation of plant parameters, inaccuracy arising from identification, etc. Since aerodynamic parameters are functions of the flight condition, a missile is a nonlinear and rapidly parameter-varying plant due to operation in a large range of aerodynamic conditions. The aerodynamic parameters exist in state matrix and input matrix [1]-[2], and are uncertain in practice. According to this, missile control problem can be formulated into a problem of stability of robust control systems subject to uncertainties with respect to both state and input.

In this paper, the stabilization control problem for a class of systems subject to nonparametric time-varying uncertainties with respect to both state and input is investigated. Before introducing our main idea, some accepted control methods in the literature to handle the uncertainties are briefly reviewed.

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A direct way is to estimate all of the unknown parameters, then use them simultaneously. Lyapunov methods are used to analyze the stability of closed-loop system. In [3], $\mathcal{L}_1$ adaptive control architecture is proposed for systems in the presence of unknown input gain, time-varying unknown parameters and disturbances. This controller leads to uniform performance bounds for the system's input and output signals, which can be systematically improved by increasing the adaptation rate. In [4], the problem of a planar rigid body, with unknown rotational inertia and an unknown input nonlinearity, tracking a desired angular velocity trajectory is addressed using adaptive feedback control. By parameterizing the input nonlinearity, an adaptive feedback linearization controller is developed. In [5], two asymptotic tracking controllers are designed for output tracking of an aircraft system in the presence of parametric uncertainty and unknown, nonlinear disturbances, which are not linearly parameterizable. An adaptive extension is then presented, in which feedforward adaptive estimates of the input uncertainty are used. In [3]-[5], an adaptive controller may require many integrators corresponding to many unknown parameters for an uncertain system because each unknown parameter needs an integrator to estimate. This will lead to a resulting closed-loop system with a reduced stability margin. In addition, the estimates may not approach the true parameters without persistently exciting signals, which are difficult to generate in practice especially when the number of unknown parameters is large [6, pp. 111-118]. The second way is to design an inverse control by a neural network to cancel the input nonlinearities in order to obtain a linear function [7],[8]. Contrary to the traditional inverse control schemes, a neural network is employed to approximate an unknown nonlinear term instead in [9]. In fact, the neural network methods can also be considered as an adaptive control method just with different basis functions. So, this way also has the same problems as that for adaptive control. The third way is to adopt sliding mode control due to its inherent advantages of fast response and insensitivity to plant parameter variation and/or external perturbation. In [10], a new sliding mode control law based on the measurability of all of the system states is presented to ensure the global reaching condition of the sliding mode for systems subject to uncertainties with respect to both state and input. Based on it, an output feedback controller was further proposed in [11]. Whereas, sliding mode controllers essentially rely on infinite gains to achieve good tracking performance, which is not always feasible in practice. The drawbacks of the high-gain feedback solutions are related to the fact that they may saturate the joint actuators and excite high-frequency modes, etc.

In order to overcome these drawbacks, this paper proposes a novel control, namely additive-state-decomposition-based dynamic inversion stabilized control, which is used to stabilize a class of systems subject to nonparametric time-varying uncertainties with respect to both state and input. Unlike many literatures, uncertainties are nonparametric and time-varying. Therefore, the existing methods, namely adaptive control method and neural network method, are not applicable directly. The key idea behind the proposed method is to lump nonparametric time-varying uncertainties with respect to both state

and input into one disturbance by additive state decomposition, and then to compensate for it. The proposed controller is continuous and can produce asymptotic stability if the uncertainties are time invariant. At this point, it overcomes the drawbacks of sliding mode controllers. Finally, the additive-state-decomposition-based dynamic inversion stabilized controller is applied to tracking control for the missile's angle of attack. Unlike the control methods proposed in [1]-[2], less assumptions are imposed on the time-varying term. Furthermore, the design of the controller is easier and the structure is simpler.

II. Problem Formulation and Additive State Decomposition

A. Problem Formulation

Consider a class of systems subject to nonparametric time-varying uncertainties with respect to both state and input as follows:

$$\dot{x}(t) = A_0 x(t) + b(h(t,u) + \sigma(t,x)), x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state (measurable), $u(t) \in \mathbb{R}$ is the control, $b \in \mathbb{R}^n$ is known constant vector, $A_0 \in \mathbb{R}^{n\times n}$ is a known matrix, $h : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is an unknown nonlinear function, and $\sigma : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is an unknown nonlinear time-varying disturbance. For system (1), the following assumptions are made.

Assumption 1. The pair (A_0, b) is controllable.

Assumption 2. The unknown nonlinear function h satisfies $h(t,0) \equiv 0,\ \left|\frac{\partial h}{\partial t}\right| \le l_{h_t}|u|,\ \frac{\partial h}{\partial u} > \underline{l}_{h_u}$ and $\left|\frac{\partial h}{\partial u}\right| \le \bar{l}_{h_u}$, $\forall u \in \mathbb{R}$, $\forall t \ge 0$, where $l_{h_t}, \underline{l}_{h_u}, \bar{l}_{h_u} > 0$.

Assumption 3. The time-varying disturbance $\sigma(t,x)$ satisfies $|\sigma(t,x)| \le k_\sigma \|x(t)\| + \delta_\sigma(t)$, $\left\|\frac{\partial \sigma}{\partial x}\right\| \le l_{\sigma_x}$ and $\left|\frac{\partial \sigma}{\partial t}\right| \le l_{\sigma_t}\|x(t)\| + d_\sigma(t)$, where $\frac{\partial \sigma}{\partial x} \triangleq [\frac{\partial \sigma}{\partial x_1} \cdots \frac{\partial \sigma}{\partial x_n}] \in \mathbb{R}^{1\times n}$, $k_\sigma, \delta_\sigma(t), l_{\sigma_x}, l_{\sigma_t}, d_\sigma(t) > 0$, $\forall t \ge 0$, and $\delta_\sigma(t), d_\sigma(t)$ are bounded.

Remark 1. If h is a dead zone function such as $h(t,u) = \begin{cases} u & |u| \ge \mu \\ 0 & |u| < \mu \end{cases}$, then it can be reformulated as $h(t,u) = u + \delta(t)$, where $|\delta(t)| \le \mu$. In practice, we do not need to know the parameters $l_{h_t}, \underline{l}_{h_u}, \bar{l}_{h_u}, k_\sigma, \delta_\sigma, l_{\sigma_x}, l_{\sigma_t}, d_\sigma$.

The control objective is to design a stabilized controller to drive the system state such that $x(t) \to 0$ as $t \to \infty$ or it is ultimately bounded by a small value. In the following, for convenience, we will drop the notation t except when necessary for clarity.

B. Additive State Decomposition

In order to make the paper self-contained, additive state decomposition in [12],[13] is recalled briefly here. Consider the following 'original' system:

$$f(t, \dot{x}, x) = 0, x(0) = x_0 \tag{2}$$

where $x \in \mathbb{R}^n$. We first bring in a 'primary' system having the same dimension as (2), according to:

$$f_p(t, \dot{x}_p, x_p) = 0, x_p(0) = x_{p,0} \tag{3}$$

where $x_p \in \mathbb{R}^n$. From the original system (2) and the primary system (3) we derive the following 'secondary' system:

$$f(t, \dot{x}, x) - f_p(t, \dot{x}_p, x_p) = 0, x(0) = x_0 \tag{4}$$

where $x_p \in \mathbb{R}^n$ is given by the primary system (3). Define a new variable $x_s \in \mathbb{R}^n$ as follows:

$$x_s \triangleq x - x_p. \tag{5}$$

Then the secondary system (4) can be further written as follows:

$$f(t, \dot{x}_s + \dot{x}_p, x_s + x_p) - f_p(t, \dot{x}_p, x_p) = 0, x_s(0) = x_0 - x_{p,0}. \tag{6}$$

From the definition (5), we have

$$x(t) = x_p(t) + x_s(t), t \geq 0. \tag{7}$$

Remark 2. By the additive state decomposition, the system (2) is decomposed into two subsystems with the same dimension as the original system. In this sense our decomposition is "additive". In addition, this decomposition is with respect to state. So, we call it "additive state decomposition".

As a special case of (2), a class of differential dynamic systems is considered as follows:

$$\begin{aligned} \dot{x} &= f(t, x), x(0) = x_0, \\ y &= g(t, x) \end{aligned} \tag{8}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Two systems, denoted by the primary system and (derived) secondary system respectively, are defined as follows:

$$\begin{aligned} \dot{x}_p &= f_p(t, x_p), x_p(0) = x_{p,0} \\ y_p &= g_p(t, x_p) \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dot{x}_s &= f(t, x_p + x_s) - f_p(t, x_p), x_s(0) = x_0 - x_{p,0}, \\ y_s &= g(t, x_p + x_s) - g_p(t, x_p) \end{aligned} \tag{10}$$

where $x_s \triangleq x - x_p$ and $y_s \triangleq y - y_p$. The secondary system (10) is determined by the original system (8) and the primary system (9). From the definition, we have

$$x(t) = x_p(t) + x_s(t), y(t) = y_p(t) + y_s(t), t \geq 0. \tag{11}$$

III. Additive-State-Decomposition-Based Dynamic Inversion Stabilized Control

In this section, the considered uncertain system is first transformed into a determinate system but subject to a lumped disturbance by additive state decomposition. Then a dynamic inversion method is applied to this transformed system. Finally, the performance of the resulting closed-loop system is analyzed.

A. Decomposition

Since the pairs (A_0, b) is controllable by *Assumption 1*, we can always find a vector $k \in \mathbb{R}^n$ such that $A_0 + bk^T$ is stable. This also implies that there exist $0 < P, M \in \mathbb{R}^{n\times n}$ such that

$$P\left(A_0 + bk^T\right) + \left(A_0 + bk^T\right)^T P = -M. \tag{12}$$

According to this, we rewrite the system (1) to be

$$\dot{x} = Ax + b\left(h\left(t,u\right) - k^T x + \sigma\left(t,x\right)\right), x\left(0\right) = x_0 \tag{13}$$

where $A = A_0 + bk^T$. Consider the system (13) as the original system. We choose the primary system as follows:

$$\dot{x}_p = Ax_p + bu, x_p\left(0\right) = 0. \tag{14}$$

Then the secondary system is determined by the original system (13) and the primary system (14) with the rule (10), and we obtain that

$$\dot{x}_s = Ax_s + b\left(-u + h\left(t,u\right) - k^T\left(x_p + x_s\right) + \sigma\left(t, x_p + x_s\right)\right), x_s\left(0\right) = x_0. \tag{15}$$

According to (11), we have

$$x = x_p + x_s. \tag{16}$$

The secondary system is further written as

$$\dot{x}_s = Ax_s + b\left(-u + h\left(t,u\right) - k^T x + \sigma\left(t,x\right)\right), x_s\left(0\right) = x_0. \tag{17}$$

Before proceeding further with the development of this paper, the following preliminary result is needed.

Theorem 1. Suppose $\det(sI_n - A) = (\alpha s + 1)\left(\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0\right)$. If *Assumption 1* holds and $(sI_n - A)^{-1}b$ is asymptotically stable, then there exists a $c \in \mathbb{R}^n$ such that

$$c^T(sI_n - A)^{-1}b = \frac{1}{\alpha s + 1},$$

where $\alpha > 0$, $c = \left(N^{-1}\right)^T \bar{c} \in \mathbb{R}^n$, $\bar{c} = \begin{bmatrix} \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{bmatrix}^T$, and the matrix $N \in \mathbb{R}^{n\times n}$ is defined that adj$(sI_n - A)b = N[s^{n-1} \ \cdots \ 1]^T$.

Proof. Given a vector $c \in \mathbb{R}^n$, we have

$$c^T(sI_n - A)^{-1}b = \frac{c^T \text{adj}(sI_n - A)b}{\det(sI_n - A)}$$
$$= \frac{c^T N[s^{n-1} \cdots 1]^T}{(\alpha s + 1)\left(\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0\right)}.$$

If the pair (A_0, b) is controllable by *Assumption 1*, then the pair (A, b) is controllable as well. Consequently, the matrix N is nonsingular [14, Lemma 3]. Let $c = \left(N^{-1}\right)^T \bar{c} \in \mathbb{R}^n$, $\bar{c} = \begin{bmatrix} \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{bmatrix}^T \in \mathbb{R}^n$. Then

$$c^T(sI_n - A)^{-1}b = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{(\alpha s + 1)\left(\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0\right)}.$$

Since $(sI_n - A)^{-1}b$ is asymptotically stable, $\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0$ is an asymptotically stable $(n-1)$th order polynomial. So, we have $c^T(sI_n - A)^{-1}b = \frac{1}{\alpha s+1}$. $\square$

Theorem 1 shows that we can always find a $c \in \mathbb{R}^n$ such that $G = \frac{1}{\alpha s+1}$. According to *Theorem 1*, we define a new output $y = c^T x$ and rearrange (14)-(16) to be

$$\dot{x}_p = Ax_p + bu, x_p(0) = 0$$
$$y = c^T x_p + d_l \tag{18}$$

where $d_l = c^T x_s = G\left(-u + h(t,u) - k^T x + \sigma(t,x)\right) + c^T e^{At} x_0$ is called the lumped disturbance. Furthermore, (18) is written as

$$y = Gu + d_l. \tag{19}$$

The lumped disturbance d_l includes uncertainties, disturbance and input. Fortunately, since $y_p = c^T x_p = Gu$ and the output y are known, the lumped disturbance d_l can be observed by

$$\hat{d}_l = y - Gu.$$

It is easy to see that $\hat{d}_l \equiv d_l$.

B. *Dynamic Inversion Control*

So far, by the additive state decomposition, the uncertain system (1) has been transformed into a determinate system (19) but subject to a lumped disturbance, which is shown in Fig.1.

For the system (19), since G is minimum phase and known, the dynamic inversion tracking controller design is represented as follows:

$$u = -G^{-1}\hat{d}_l. \tag{20}$$

Substituting (20) into (19) results in

$$y = -GG^{-1}\hat{d}_l + d_l$$
$$= -\hat{d}_l + d_l = 0$$

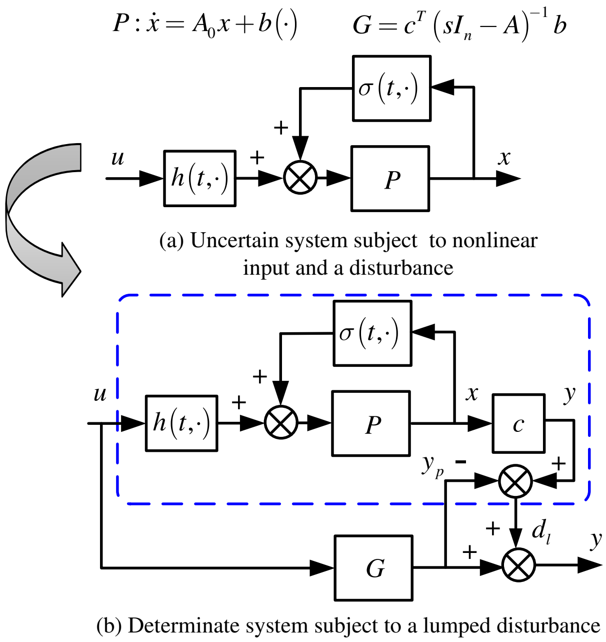

Fig. 1. Model Transformation

where $\hat{d} \equiv d_l$ is utilized. As a result, the perfect tracking is achieved. However, the proposed controller (20) cannot be realizable. By introducing a low-pass filter Q, the controller (20) is modified as follows:

$$u = -QG^{-1}\hat{d}_l. \tag{21}$$

By employing the controller (21), the output becomes

$$y = (1 - Q)\,\hat{d}_l. \tag{22}$$

Since Q is a low-pass filter and the low-frequency range is often dominant in a signal, it is expected that the output will be attenuated by the transfer function $1 - Q$. However, the mention above is only a straightforward explanation. A detailed analysis is given in the following section.

C. Performance Analysis

Since the lumped disturbance d_l involves the input u, the resulting closed-loop system may be instable. Next, some conditions are given to ensure that the control input u is bounded. Substituting $\hat{d}_l$ into (21) results into

$$\begin{aligned} u &= -QG^{-1}\left[G\left(-u + h\left(t,u\right) - k^T x + \sigma\left(t,x\right)\right) + c^T e^{At} x_0\right] \\ &= Q\left(u - h\left(t,u\right) + k^T x - \sigma\left(t,x\right)\right) - QG^{-1}c^T e^{At} x_0. \end{aligned} \tag{23}$$

Multiplying Q^{-1} on both sides of (23) yields

$$Q^{-1}u = u - h\left(t,u\right) + k^T x - \sigma\left(t,x\right) + \xi \tag{24}$$

where $\xi = -G^{-1}c^T e^{At}x_0$. Since $G = \frac{1}{\alpha s+1}$ and A is stable, the term ξ will tend to zero exponentially. A simple way is to choose $Q = \frac{1}{\epsilon s+1}$, where $\epsilon > 0$ can be considered as a singular perturbation parameter. By the filter Q, (24) is further written as

$$\epsilon \dot{u} = -h\left(t,u\right) + k^T x - \sigma\left(t,x\right) + \xi. \tag{25}$$

So far, the dynamics (13) and (25) form a new closed-loop dynamics. In geometric singular perturbation theory [15], the behavior of singularly perturbed systems is determined using geometric constructs of the reduced-order models, which are obtained by substituting $\epsilon = 0$ in (25). According to this, we have

$$0 \approx -h\left(t,u\right) + k^T x - \sigma\left(t,x\right) + \xi.$$

By this result, (13) becomes

$$\dot{x} \approx Ax + b\xi, x\left(0\right) = x_0.$$

The dynamics above imply that $x \to 0$ as $t \to \infty$ if A is stable. Furthermore, u is bounded by (25). The following theorem will give an explicit bound on ϵ which can ensure the stability of closed-loop dynamics forming by (13) and (25).

Theorem 2. Suppose i) *Assumptions 1-3* hold, ii) the controller is designed as (21) with $Q = \frac{1}{\epsilon s+1}$, iii) ϵ satisfies

$$\epsilon < \frac{\underline{l}_{h_u}}{\gamma_1 + \frac{2}{\gamma_0}\left(\gamma_2 + l_{\sigma_t}\right)^2} \tag{26}$$

where

$$\begin{aligned}
\gamma_0 &= \lambda_{\min}\left(M\right) \\
\gamma_1 &= 2\left(\|k\| + l_{\sigma_x}\right)\|b\| + 2\frac{l_{h_t}}{\underline{l}_{h_u}} \\
\gamma_2 &= \|P\|\|b\| + \|A\|\left(\|k\| + l_{\sigma_x}\right) + \|k\| + k_\sigma.
\end{aligned} \tag{27}$$

Then the state of system (1) is uniformly ultimately bounded with respect to the bound $\sqrt{\frac{1}{\eta(\epsilon)}\frac{\epsilon}{\underline{l}_{h_u}}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma\right)$. Furthermore, if $l_{h_t}\delta_\sigma \to 0$ and $d_\sigma \to 0$, then the state $x \to 0$ as $t \to \infty$. In addition, if ϵ is sufficiently small, then the uniformly ultimate bound of state x is $\sqrt{\frac{2\lambda_{\max}(P)}{\gamma_0}\frac{\epsilon}{\underline{l}_{h_u}}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma\right)$.

Proof. See Appendix.

Remark 3. According to *Theorem 2*, if the uncertainties are time invariant, namely $l_{h_t} = 0$ and $d_\sigma = 0$, then the proposed controller can produce asymptotic stability. Also, from *Theorem 2*, a sufficiently small ϵ will satisfy (26) and the ultimate bound will be reduced by decreasing ϵ. However, we should point out that a small ϵ in turn will result in a reduced robustness of the closed-loop system. For example, consider a simple situation where $h\left(t,u\right) = u\left(t-\tau\right)$. Given a $\tau > 0$, the dynamical system $\epsilon\dot{u} = -u\left(t-\tau\right)$ will loss stability by choosing a sufficiently small ϵ no matter how small the

delay τ is[1]. Therefore, an appropriate $\epsilon > 0$ should be chosen to achieve a tradeoff between tracking performance and robustness. We do not need to know the parameters $l_{h_t}, \underline{l}_{h_u}, \bar{l}_{h_u}, k_\sigma, \delta_\sigma, l_{\sigma_x}, l_{\sigma_t}, d_\sigma$ and choose ϵ according to practical condition.

From the above analysis, the design procedure is summarized as follows.

1. Design a state feedback gain $k \in \mathbb{R}^n$ such that $A = A_0 + bk^T$ is stable with $\det(sI_n - A) = (\alpha s + 1)\left(\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0\right)$.

2. Design an output matrix $c = \left(N^{-1}\right)^T \bar{c} \in \mathbb{R}^n$ where $\text{adj}(sI_n - A)b = N[s^{n-1} \cdots 1]^T$ and $\bar{c} = \left[\begin{array}{cccc} \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{array}\right]^T \in \mathbb{R}^n$.

3. Design (21) with $G = \frac{1}{\alpha s+1}$ and $Q = \frac{1}{\epsilon s+1}$.

4. Choose an appropriate $\epsilon > 0$ in practice.

IV. An Application to Missile Longitudinal Tracking

The following linear parameter-varying system approximates a missile's longitudinal axis dynamics [1]-[2]:

$$\left[\begin{array}{c} \dot{\alpha}(t) \\ \dot{q}(t) \end{array}\right] = \left[\begin{array}{cc} -Z_\alpha(t) & 1 \\ -M_\alpha(t) & 0 \end{array}\right]\left[\begin{array}{c} \alpha(t) \\ q(t) \end{array}\right] + \left[\begin{array}{c} 0 \\ M_\delta(t) \end{array}\right](\delta_m(t) + d(t)) \tag{28}$$

where α is the angle of attack, q is the pitch rate, and δ_m is the fin deflection. We assume that α, q are measurable and $Z_\alpha(t)$ is known[2]. In the following we will formulate (28) as (1). The objective is to let $\alpha(t) \to r(t)$ as $t \to \infty$, where $r(t)$ is known and twice continuously differentiable.

A. Problem Formulation

Define

$$x = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right], x_1 = \alpha - r, x_2 = -Z_\alpha \alpha + q - \dot{r}.$$

We have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dot{Z}_\alpha \alpha - Z_\alpha \dot{\alpha} + \dot{q} - \ddot{r} \\ &= -\dot{Z}_\alpha \alpha - Z_\alpha(-Z_\alpha \alpha + q) - M_\alpha \alpha - \ddot{r} + M_\delta(\delta_m + d). \end{aligned} \tag{29}$$

[1]The characteristic equation of $\epsilon\dot{u} = -u(t-\tau)$ is $\epsilon s + e^{-s\tau} = 0$, which can be approximated by $(\epsilon - \tau)s + 1 = 0$. Therefore, one solution of the characteristic equation is $s \approx \frac{1}{\tau - \epsilon}$. If $\epsilon < \tau$, then $s > 0$, namely the dynamical system $\epsilon\dot{u} = -u(t-\tau)$ is unstable.

[2]The requirement of an exact $Z_\alpha(t)$ is only used for the problem formulation. In practice, the parameter can be inexact. This will be shown in the following simulation.

In practice, we only know the approximation of the parameters $M_\alpha(t), M_\delta(t)$, namely $\bar{M}_\alpha(t), \bar{M}_\delta(t)$. With these information, the control δ_m is designed as

$$\delta_m = \frac{1}{\bar{M}_\delta}\left(k^T x + \dot{Z}_\alpha \alpha + Z_\alpha(-Z_\alpha \alpha + q) + \bar{M}_\alpha \alpha + \ddot{r} + u\right) \tag{30}$$

where the additional part $k^T x$ is to stabilized the closed-loop system, while the other additional part $\frac{M_\delta}{\bar{M}_\delta}\left(\dot{Z}_\alpha \alpha + Z_\alpha(-Z_\alpha \alpha + q) + \bar{M}_\alpha \alpha + \ddot{r}\right)$ is to compensate for $-\dot{Z}_\alpha \alpha - Z_\alpha(-Z_\alpha \alpha + q) - M_\alpha \alpha - \ddot{r}$. By substituting (30) into (29), the missile's longitudinal axis dynamics are formulated as (1), namely

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(h(t,u) + \sigma(t,x)) \tag{31}$$

where $h(t,u) = \frac{M_\delta}{\bar{M}_\delta} u$ and $\sigma(t,x) = \frac{M_\delta}{\bar{M}_\delta}\left(k^T x + \dot{Z}_\alpha \alpha + Z_\alpha(-Z_\alpha \alpha + q) + \bar{M}_\alpha \alpha + \ddot{r}\right) - \dot{Z}_\alpha \alpha - Z_\alpha(-Z_\alpha \alpha + q) - M_\alpha \alpha - \ddot{r} + M_\delta d$. Obviously, if $\bar{M}_\alpha = M_\alpha$ and $\bar{M}_\delta = M_\delta$, then the system above becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u + M_\delta d).$$

B. Controller Design

For the system (31), according to the design procedure at the end of Section III.C, we have the following design.

Step 1. According to the design procedure, we design $k = \begin{bmatrix} -12 & -7 \end{bmatrix}^T$ resulting in

$$A = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$$

with $\det(sI_2 - A) = \left(\frac{1}{4}s + 1\right)(4s + 12)$.

Step 2. According to A and b, we can obtain $\bar{c} = \begin{bmatrix} 4 & 12 \end{bmatrix}^T$ and $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consequently, the output matrix is designed as $c = \left(N^{-1}\right)^T \bar{c} = \begin{bmatrix} 12 & 4 \end{bmatrix}^T$.

Step 3. Design controller $u = -QG^{-1}\hat{d}_l$, where $G = \frac{1}{\frac{1}{4}s+1}$, $Q = \frac{1}{\epsilon s+1}$, $\hat{d}_l = c^T x - Gu$ and ϵ will be specified latter.

C. Numerical Simulation

Similar to [2], we choose the parameters

$$Z_\alpha(t) = 2.35 - 2.05\sin(0.05\pi t), M_\alpha(t) = 5 + 375\sin(-0.014\pi t)$$
$$M_\delta(t) = 3 - 2.5\cos(0.04\pi t), d(t) \equiv 0.1 \tag{32}$$

where the working interval $0 \le t \le 100s$. The initial conditions are $\alpha(0) = -1$ and $q(0) = 0$. Denote $F = \begin{bmatrix} -Z_\alpha(t) & 1 \\ -M_\alpha(t) & 0 \end{bmatrix}$. When $t = 0s$, eigenvalues of matrix F are $-1.1750 \pm 1.9025i$. When $t = 50s$,

eigenvalues of matrix F are ± 17.4244. The considered linear parameter-varying system is unstable during the working interval. In practice, we assume that we only know the approximation of the parameters, namely,

$$\bar{Z}_\alpha(t) = 2.2 - 2\sin(0.05\pi t), \dot{\bar{Z}}_\alpha(t) \approx 0$$
$$\bar{M}_\alpha(t) = 4 + 360\sin(-0.014\pi t), \bar{M}_\delta(t) = 2.8 - 2.1\cos(0.04\pi t) \tag{33}$$

where $\bar{Z}_\alpha(t)$ is approximated to $Z_\alpha(t)$.

First, according to the known information, we design δ_m but without u, namely

$$\delta_m = \frac{1}{\bar{M}_\delta}\left(k^T x + \dot{\bar{Z}}_\alpha \alpha + \bar{Z}_\alpha\left(-\bar{Z}_\alpha \alpha + q\right) + \bar{M}_\alpha \alpha + \ddot{r}\right).$$

The simulation is shown in Fig.2. The divergence suggests that feedback stabilization and compensation with approximated information may not achieve the tracking task, especially when the parameters vary in a large range. Let us add the control term u into δ_m with $\epsilon = 0.06$. The simulation is shown in Fig.3. As shown, the controller with the stabilized control term can make α track the reference r with a small tracking error in the presence of time-varying uncertainties. This is consistent with the conclusions of *Theorem 2*.

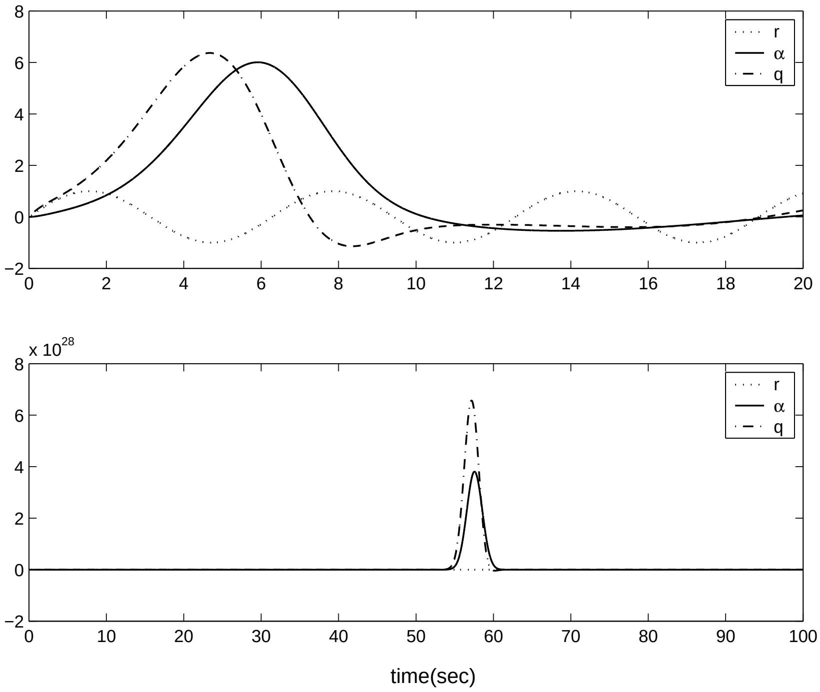

Fig. 2. Tracking performance without additive-state-decomposition-based stabilized control term

In order to show the asymptotic convergence, we assume that the tracking reference $r(t) \equiv 1$ and these time-varying parameters are frozen at $t = 50s$, namely $Z_\alpha(t) \equiv 0.3, M_\alpha(t) \equiv -298.4, M_\delta(t) \equiv$

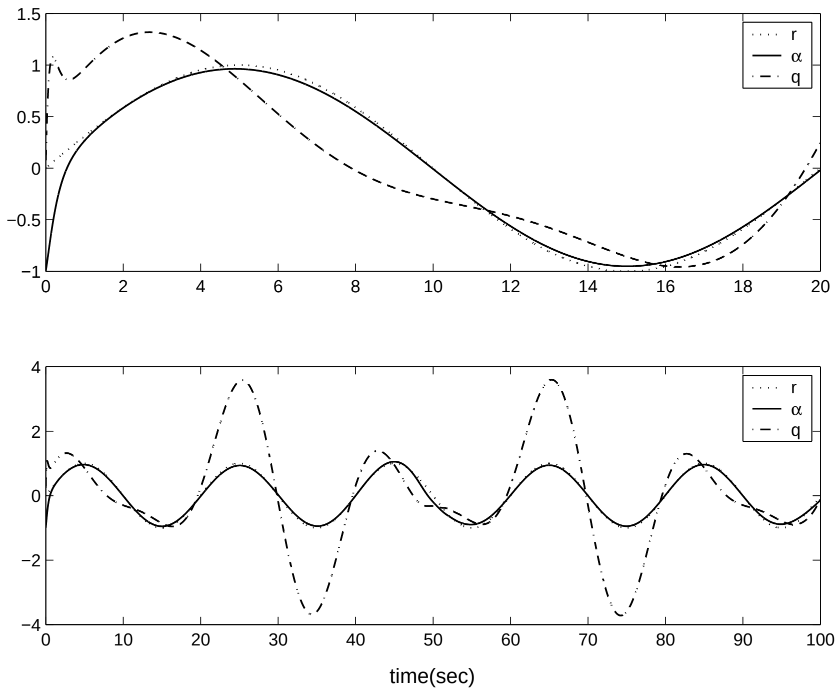

Fig. 3. Tracking performance with additive-state-decomposition-based stabilized control term in the presence of time-varying uncertain parameters.

$0.5, d(t) \equiv 0.1$. Similarly, we assume that we only know the approximation of the parameters $\bar{Z}_{\alpha}(t) \equiv 0.3, \dot{\bar{Z}}_{\alpha}(t) \approx 0, \bar{M}_{\alpha}(t) \equiv -287, \bar{M}_{\delta}(t) \equiv 0.7$. By the same controller with additive-state-decomposition-based stabilized control term with $\epsilon = 0.03$, we get the simulation result shown in Fig.4. The tracking error tends to zero. This result is consistent with the conclusions of *Theorem 2*.

V. Conclusions

Stabilized control for a class of systems subject to nonparametric time-varying uncertainties with respect to both state and input is considered. Our main contributions lie in two transformations: i) the presentation of a new decomposition scheme, called additive state decomposition, which transforms the uncertain system into an uncertainty-free system; ii) and a definition of output matrix which transforms the uncertainty-free system into a first order system. Based on these transformations, nonparametric time-varying uncertainties with respect to both state and input is lumped into a new disturbance at the output of the first order system. This can explain why the proposed method can handle these nonparametric time-varying uncertainties. The proposed control scheme has two salient features: less information of one system required and simple design procedure with less tuning parameters.

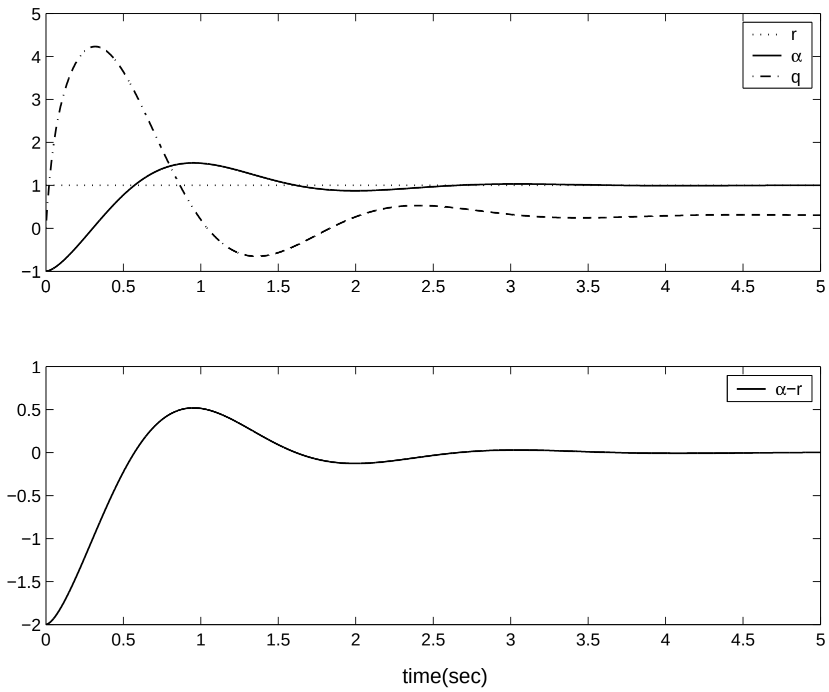

Fig. 4. Tracking performance with additive-state-decomposition-based stabilized control term in the presence of constant uncertain parameters.

VI. Appendix: Proof of Theorem 2

Denote $v = h(t, u) - k^T x + \sigma(t, x)$. Then the system (13) becomes

$$\dot{x} = Ax + bv$$

and the derivative of $\epsilon \dot{v}$ is presented as

$$\begin{aligned}
\epsilon \dot{v} &= \frac{\partial h}{\partial u}\epsilon \dot{u} + \epsilon \frac{\partial h}{\partial t} - \epsilon k^T \dot{x} + \frac{\partial \sigma}{\partial x}\epsilon \dot{x} + \epsilon \frac{\partial \sigma}{\partial t} \quad \text{((25) is used)} \\
&= \frac{\partial h}{\partial u}(-v + \xi) + \epsilon \left(-k^T + \frac{\partial \sigma}{\partial x}\right)(Ax + bv) + \epsilon \frac{\partial h}{\partial t} + \epsilon \frac{\partial \sigma}{\partial t} \\
&= \left[-\frac{\partial h}{\partial u} + \epsilon \left(-k^T + \frac{\partial \sigma}{\partial x}\right) b\right] v + \epsilon \frac{\partial h}{\partial t} + \epsilon \frac{\partial \sigma}{\partial t} + \frac{\partial h}{\partial u}\xi + \epsilon \left(-k^T + \frac{\partial \sigma}{\partial x}\right) Ax.
\end{aligned}$$

Consequently, the closed-loop dynamics (25) and (13) are

$$\begin{aligned}
\dot{x} &= Ax + bv \\
\dot{v} &= \left[-\frac{1}{\epsilon}\frac{\partial h}{\partial u} + \left(-k^T + \frac{\partial \sigma}{\partial x}\right) b\right] v + \frac{\partial h}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\xi + \left(-k^T + \frac{\partial \sigma}{\partial x}\right) Ax. \qquad (34)
\end{aligned}$$

Design a candidate Lyapunov function as follows:

$$V = x^T P x + v^2$$

where $0 < P \in \mathbb{R}^{n\times n}$ satisfies (12). Taking the derivative of V along the solution of (34) yields

$$\begin{aligned}\dot{V} &= x^T\left(PA + A^TP\right)x + 2x^TPbv + 2v\left[-\frac{1}{\epsilon}\frac{\partial h}{\partial u} + \left(-k^T + \frac{\partial \sigma}{\partial x}\right)b\right]v \\ &\quad + 2v\left(-k^T + \frac{\partial \sigma}{\partial x}\right)Ax + 2v\left(\frac{\partial h}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\xi\right) \\ &= x^T\left(PA + A^TP\right)x + 2v\left(\frac{\partial h}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\xi\right) \\ &\quad - 2\left[\frac{1}{\epsilon}\frac{\partial h}{\partial u} - \left(-k^T + \frac{\partial \sigma}{\partial x}\right)b\right]v^2 + 2x^T\left[Pb - A^T\left(-k^T + \frac{\partial \sigma}{\partial x}\right)^T\right]v.\end{aligned}$$

By (12), we have $x^T\left(PA + A^TP\right)x \leq -\gamma_0\left\|x\right\|^2$, where $\gamma_0 = \lambda_{\min}\left(M\right)$. Then

$$\begin{aligned}\dot{V} &\leq -\gamma_0\left\|x\right\|^2 + 2\left|v\right|\left(l_{h_t}\left|u\right| + l_{\sigma_t}\left\|x\right\| + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon}\left|\xi\right|\right) \\ &\quad - 2\left(\frac{1}{\epsilon}\underline{l}_{h_u} - \frac{1}{2}\gamma_1\right)v^2 + 2\gamma_2\left|v\right|\left\|x\right\| \quad (\textit{Assumptions 2-3} \text{ are used})\end{aligned} \tag{35}$$

Next, before proceeding further, we will derive the relationship between u and v. By the mean-value theorem, we have

$$h\left(t,u\right) = h\left(t,0\right) + \left.\frac{\partial h\left(s\right)}{\partial s}\right|_{s=c}u$$

where $0 \leq c \leq u$. Furthermore, since $v = h\left(t,u\right) - k^Tx + \sigma\left(t,x\right)$, we have

$$u = \left(\left.\frac{\partial h\left(s\right)}{\partial s}\right|_{s=c}\right)^{-1}\left(v + k^Tx - \sigma\left(t,x\right) - h\left(t,0\right)\right).$$

Further by *Assumptions 2-3*, the equation above becomes

$$\begin{aligned}\left|u\right| &\leq \frac{1}{\underline{l}_{h_u}}\left(\left|v\right| + \left\|k\right\|\left\|x\right\| + \left|\sigma\left(t,x\right)\right|\right) \\ &\leq \frac{1}{\underline{l}_{h_u}}\left(\left|v\right| + \left(\left\|k\right\| + k_\sigma\right)\left\|x\right\| + \delta_\sigma\right).\end{aligned}$$

With this inequality above and relationship

$$2\left|v\right|\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon}\left|\xi\right|\right) \leq \frac{1}{\epsilon}\underline{l}_{h_u}v^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon}\left|\xi\right|\right)^2,$$

the inequality (35) becomes

$$\dot{V} \leq -\gamma_0\left\|x\right\|^2 - \left(\frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1\right)v^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon}\left|\xi\right|\right)^2 + 2\left(\gamma_2 + l_{\sigma_t}\right)\left|v\right|\left\|x\right\|.$$

Since the inequality $2ab \leq a^2 + b^2$ always holds, then

$$2\left(\gamma_2 + l_{\sigma_t}\right)\left|v\right|\left\|x\right\| \leq \frac{\gamma_0}{2}\left\|x\right\|^2 + \frac{2}{\gamma_0}\left(\gamma_2 + l_{\sigma_t}\right)^2v^2.$$

Consequently, we have

$$\dot{V} \leq -\frac{\gamma_0}{2}\left\|x\right\|^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon}\left|\xi\right|\right)^2 - \left[\frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1 - \frac{2}{\gamma_0}\left(\gamma_2 + l_{\sigma_t}\right)^2\right]v^2.$$

Furthermore, we have

$$\dot{V} \leq -\eta(\epsilon) V + \frac{\epsilon}{\underline{l}_{h_u}} \left(\frac{l_{h_t}}{\underline{l}_{h_u}} \delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon} |\xi| \right)^2 .$$

where $\eta(\epsilon) = \min(\frac{\gamma_0}{2\lambda_{\max}(P)}, \frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1 - \frac{2}{\gamma_0}(\gamma_2 + l_{\sigma_t})^2)$. If (26) is satisfied, then $\eta(\epsilon) > 0$. So, we have $V(t) \to \mathcal{B}\left(\frac{1}{\eta(\epsilon)} \frac{\epsilon}{\underline{l}_{h_u}} \left(\frac{l_{h_t}}{\underline{l}_{h_u}} \delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon} |\xi| \right)^2 \right)$ as $t \to \infty$, namely

$$\|x(t)\| \to \mathcal{B}\left(\sqrt{\frac{\epsilon}{\lambda_{\min}(P)\eta(\epsilon)\underline{l}_{h_u}}} \left(\frac{l_{h_t}}{\underline{l}_{h_u}} \delta_\sigma + d_\sigma + \frac{\bar{l}_{h_u}}{\epsilon} |\xi| \right) \right)$$

as $t \to \infty$, where $\mathcal{B}(\delta) \triangleq \{\xi \in \mathbb{R} \,|\, \|\xi\| \leq \delta\}$, $\delta > 0$. The notation $x(t) \to \mathcal{B}(\delta)$ means $\min_{y \in \mathcal{B}(\delta)} |x(t) - y| \to 0$ as $t \to \infty$. Since $\xi \to 0$ as $t \to \infty$, $\|x(t)\| \to \mathcal{B}\left(\sqrt{\frac{\epsilon}{\lambda_{\min}(P)\eta(\epsilon)\underline{l}_{h_u}}} \left(\frac{l_{h_t}}{\underline{l}_{h_u}} \delta_\sigma + d_\sigma \right) \right)$. Furthermore, if $l_{h_t}\delta_\sigma \to 0$ and $d_\sigma \to 0$, then the state $x \to 0$ as $t \to \infty$. In addition, if ϵ is sufficiently small, then $\eta(\epsilon) = \frac{\gamma_0}{2\lambda_{\max}(P)}$, namely the uniformly ultimate bound of state x is $\sqrt{\frac{2\lambda_{\max}(P)}{\gamma_0} \frac{\epsilon}{\underline{l}_{h_u}}} \left(\frac{l_{h_t}}{\underline{l}_{h_u}} \delta_\sigma + d_\sigma \right)$.

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